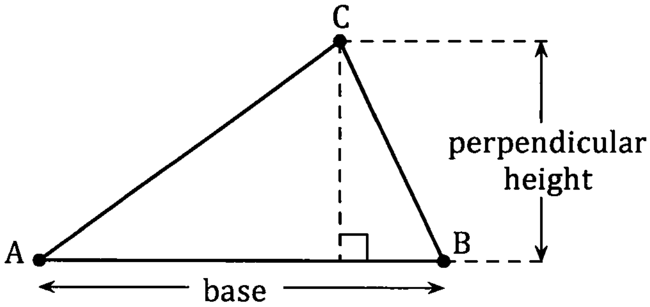
Chapter Eleven.

Trigonometry for triangles that are not right angled.

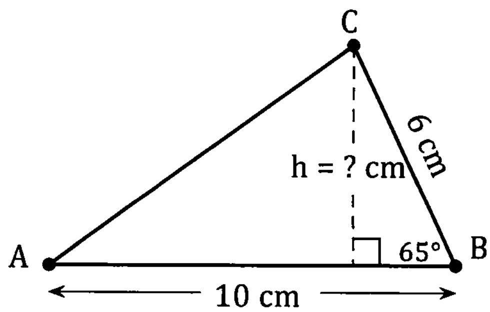
Area of a triangle.

From unit one of this course, and from earlier years, you are already familiar with the formula for the area of a triangle:



$$\text{Area of a } \Delta = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

Suppose instead you are asked to determine the area of a triangle for which you are <u>not</u> told the perpendicular height but instead know two sides and the angle between them, as shown on the right. How would you proceed then?



Exercise 11A

1. Find h in triangle ABC shown above and hence determine the area of the triangle.

Find the areas of each of the following triangles (not drawn to scale).

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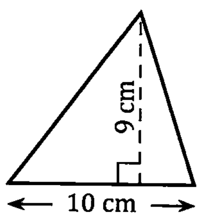

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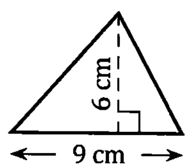

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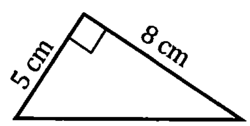

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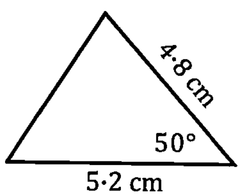

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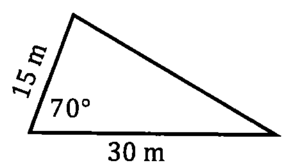

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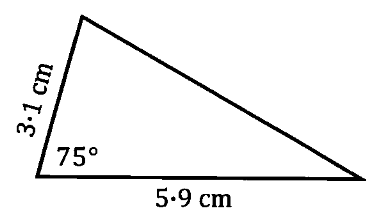

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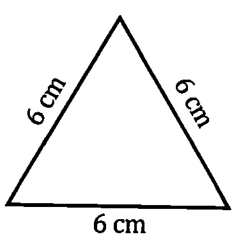

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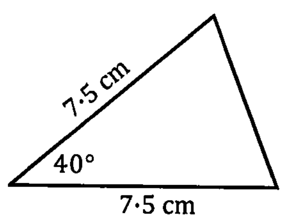

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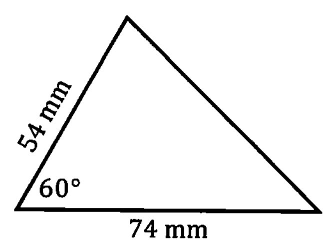
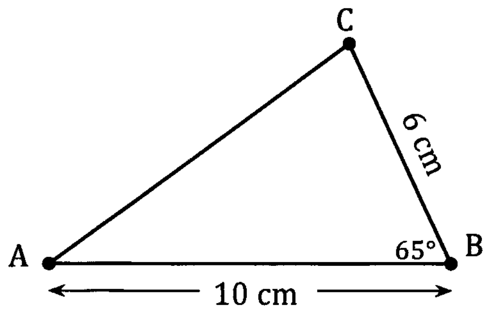
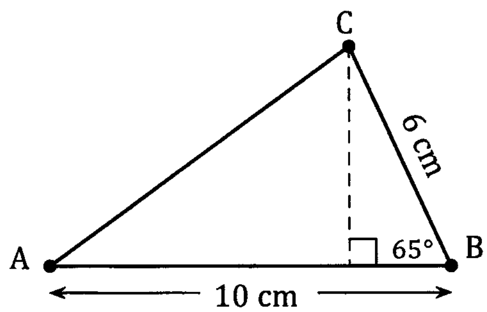


Triangles that are not right angled.

On the previous page we were asked, amongst other things, to determine the area of ΔABC, given the lengths of BA and BC and the size of ∠ABC. (i.e. given two sides and the angle between them.)



By drawing the perpendicular from C to AB we obtain right triangles. This allows trigonometry to be used to determine the height, and hence the area, of ΔABC.

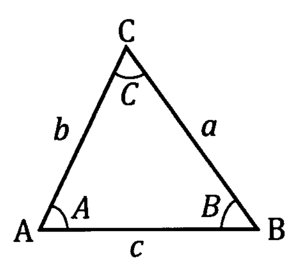


(Alternatively ΔABC could be drawn accurately from the given information and the perpendicular height could be measured.)

This approach of drawing the perpendicular from one vertex to the opposite side allows trigonometry to be used for a triangle that is not right angled. We will use this approach in this chapter to obtain three formulae that are useful when dealing with triangles that are not right angled.

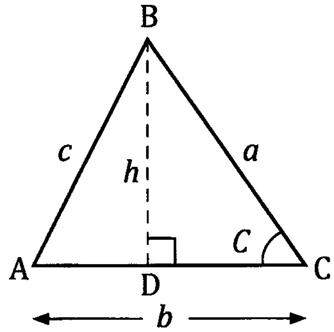
We will consider:
- ☞ a formula for the area of a triangle,
- ☞ the sine rule formula,
- ☞ the cosine rule formula.

Note: In obtaining these formulae we will use the usual convention for naming the sides and angles of a triangle. i.e. in triangle ABC the three angles are labelled A, B and C according to their vertex and the sides opposite these angles are labelled a, b and c respectively.



Area of a triangle given two sides and the included angle.

Consider ΔABC with the perpendicular from B to AC meeting AC at D (see diagram).



In ΔBDC, $\sin C = \frac{h}{a}$

Multiplying by a to isolate h, $h = a \sin C$

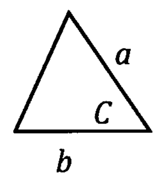
Using Area of triangle $= \frac{1}{2}$ base × height

we have Area of triangle $= \frac{1}{2} \times b \times a \sin C$

Thus:

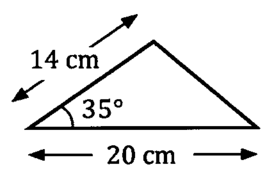
Area of a triangle $= \dfrac{ab \sin C}{2}$

i.e. the area of a triangle is half the product of two sides multiplied by the sine of the angle between them.



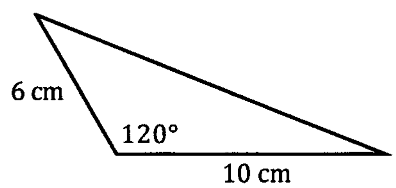
Example 1

Find the area of the triangle shown sketched on the right.



$$\text{Area} = \frac{20 \times 14 \times \sin 35°}{2}$$
$$\approx 80{\cdot}3 \text{ cm}^2$$

Now that we are dealing with any triangle, not just right triangles, we could have an obtuse angle between the two sides of known length, as shown below.



Applying our formula will involve the sine of an obtuse angle:

$$\text{Area} = \frac{10 \times 6 \times \sin 120°}{2}$$

How does your calculator respond when asked for sin 120° or sin 130° or sin 140° ... ?

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Did you notice that

$\sin 120° = \sin 60°$
$\sin 130° = \sin 50°$
$\sin 140° = \sin 40°$

in fact, to generalize: $\sin(180° - C) = \sin C$

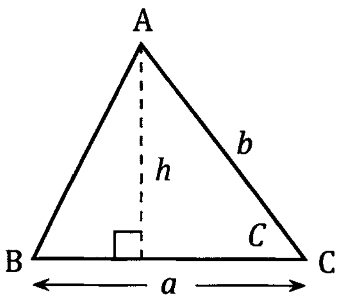
sin 120	0.8660254038
sin 60	0.8660254038
sin 130	0.7660444431
sin 50	0.7660444431

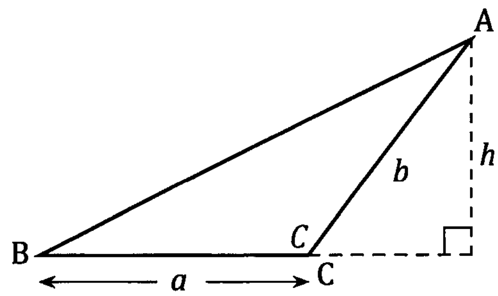
This fact means that we can use our area formula,

$$A = \frac{1}{2}ab \sin C,$$

for all triangles, even those for which angle C is obtuse.

Consider the acute angled ΔABC below left and the obtuse angled ΔABC below right.





$$\begin{aligned} \sin C &= \frac{h}{b} \\ \therefore \quad h &= b \sin C \\ \text{Area} &= \frac{1}{2} a \times h \\ &= \frac{1}{2} ab \sin C \end{aligned}$$

$$\begin{aligned} \sin(180° - C) &= \frac{h}{b} \\ \therefore \quad h &= b \sin(180° - C) \\ \text{Area} &= \frac{1}{2} a \times h \\ &= \frac{1}{2} ab \sin(180° - C) \\ &= \frac{1}{2} ab \sin C \end{aligned}$$

Thus for all triangles our area formula $A = \frac{1}{2}ab \sin C$ applies.

However, the fact that sin(180° - C)= sin C does present a difficulty if a question gives the area of a triangle and the lengths of two sides of the triangle and asks for the size of the angle between the two sides of known length. Which answer do we give – the acute angle or the obtuse angle?

For example, suppose ΔABC has an area of 3 cm^2 and is such that a = 4 cm and b = 3 cm.

Using $\quad \text{Area} = \frac{1}{2}ab\sin C$

$$3 = \frac{1}{2} \times 4 \times 3 \sin C$$

$$3 = 6 \sin C$$

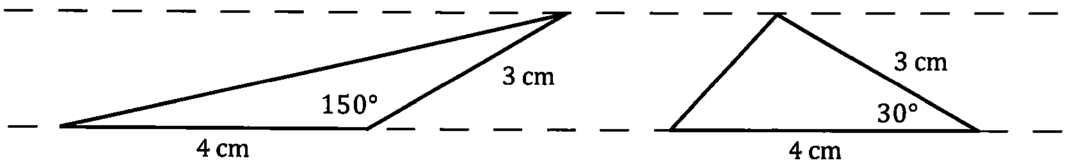
and so $\quad \sin C = 0{\cdot}5$

Now comes our dilemma:

Does C = 30° or does C = 150° ?

sin 150
0.5
sin 30
0.5

The dilemma is genuine because for the information we are given there <u>are</u> two possible triangles that "fit the facts":



The two triangles each have a base of 4 cm and are the same height as each other. Hence their areas will indeed be equal, and in this case each equal to 3 cm^2.

The information we have been given about triangle ABC is said to be *ambiguous.* (The word ambiguous meaning open to more than one interpretation.) However, there is no need to panic. In this unit we will be given sufficient information for such ambiguity to be avoided, as in the next example.

Example 2

Triangle DEF is such that e = 16 cm, d = 12 cm, the area of the triangle is 78 cm^2 and ∠DFE is an acute angle. Find the size of ∠DFE giving your answer to the nearest 0·1°.

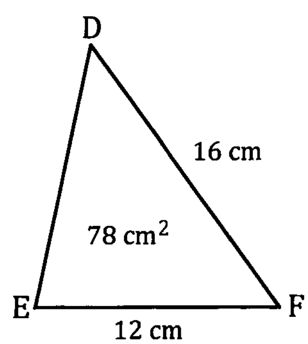
Using $\quad \text{Area} = \frac{1}{2}ab\sin C$

$$78 = \frac{1}{2} \times 12 \times 16 \sin \angle DFE$$

$\therefore \quad \sin \angle DFE = 0{\cdot}8125$

and so, given that ∠DFE is an acute angle, using a calculator:

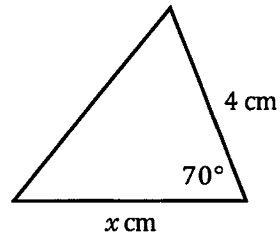
$$\angle DFE = 54{\cdot}3° \text{ (to nearest } 0{\cdot}1°)$$



(If instead we had been told that ∠DFE was an obtuse angle our final answer would have been 125·7°, i.e. 180° – 54·3°. Had we not been told anything about ∠DFE then two possible triangles would exist that each satisfied the given facts.)

Example 3

Given that the triangle sketched on the right has an area of 7 cm^2 find x correct to one decimal place.



$$\text{Area} = \tfrac{1}{2}\,(x)\;4\sin 70°$$

$$\therefore \quad 7 = \tfrac{1}{2}\,(x)\;4\sin 70°$$

$$\text{i.e.} \quad 7 = 2\,x\sin 70°$$

Solving this equation gives $x = 3{\cdot}7$, correct to 1 d.p.

Note: It is also possible to determine the area of a triangle, given the lengths of the three sides of the triangle, using a result known as **Heron's "s" formula**:

$$\text{Area of } \Delta\text{ABC} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)} \qquad \text{where} \qquad s = \frac{a+b+c}{2}\;.$$

One of the questions of a later exercise in this chapter, and one of the questions in a later Miscellaneous Exercise, reminds you of this formula and requires you to use it.

Exercise 11B

Find the area of each triangle in questions 1 to 11, giving your answers in square centimetres and correct to one decimal place. (Diagrams not necessarily to scale).

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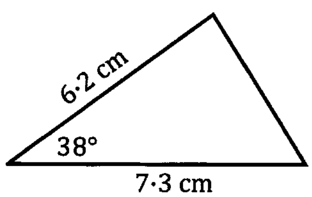

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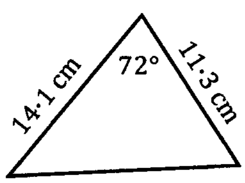

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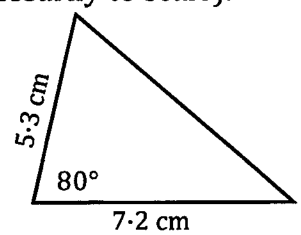

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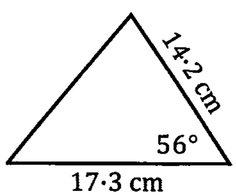

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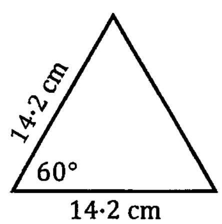

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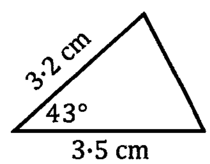

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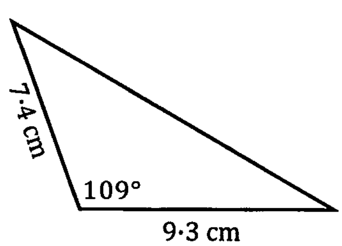

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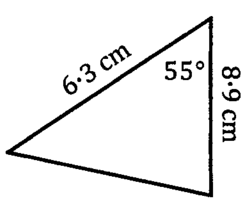

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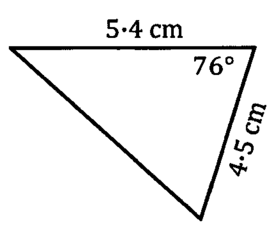
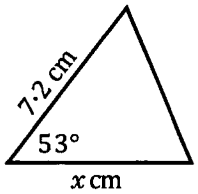


10. ΔABC given that AB = 8 cm, BC = 7 cm, AC = 5 cm and $\angle$BAC = 60°.

11. ΔPQR given that PQ = 7 cm, PR = 8 cm, RQ = 3 cm and $\angle$PRQ = 60°.

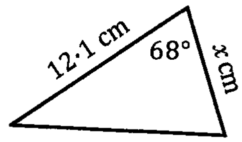
Find the value of x in each of the following, correct to one decimal place, given that the area of each triangle is as stated. (Diagrams not necessarily drawn to scale.)

12.



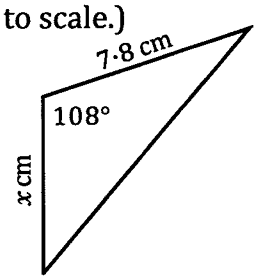
Area = 19·6 cm^2

13.



Area = 40·9 cm^2

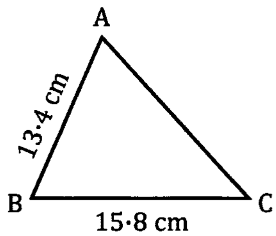
14.



Area = 24·5 cm^2

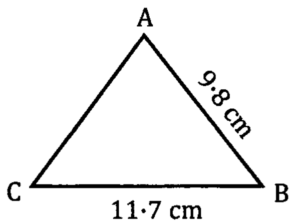
Find the size of ∠ABC in each of the following, correct to the nearest degree, given that each triangle is acute angled and the area of each triangle is as stated. (The diagrams are not necessarily drawn to scale.)

15.



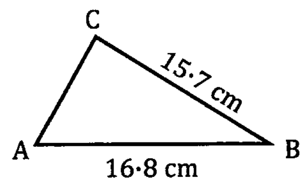
Area = 97·4 cm^2

16.



Area = 45·2 cm^2

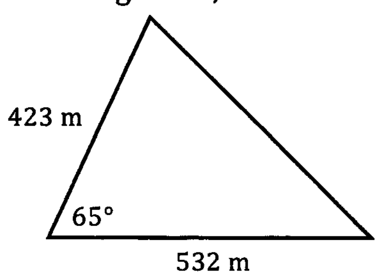
17.



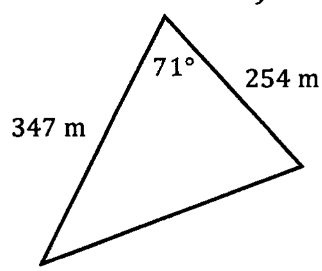
Area = 69·9 cm^2

18. If farming land in a particular region costs \$12 300 per hectare find the cost of each of the following areas, to the nearest \$1000. (1 hectare = 10 000 m^2.)

(a)

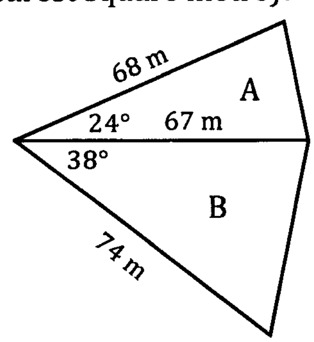


(b)

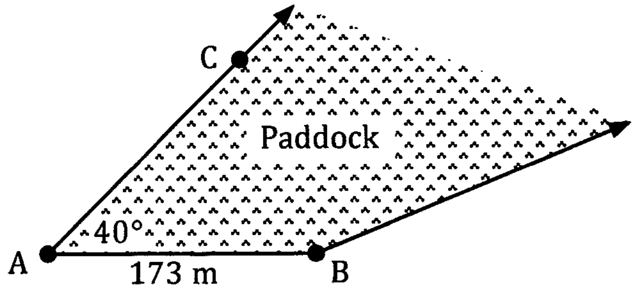


19. A triangular block of land has two sides of lengths 45 m and 30 m and the angle included between them is 70°.
A second triangular block has two sides of lengths 48 m and 35 m and the angle included between them is 50°.
Which block has the greater area and by how much (to the nearest square metre)?

20. The owners of two neighbouring triangular blocks of land, shown as A and B in the diagram on the right, are offered a total of \$1 250 000 by a property developer for the two blocks together. If they were to accept this offer and divide the money between them in the ratio of the land areas of the blocks how much would each owner receive?

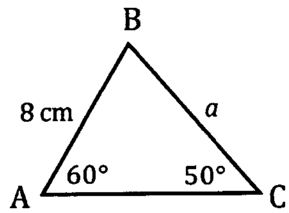


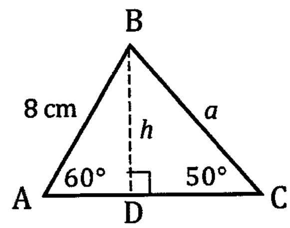
21. (Note: A square of side 100 metres has an area of 1 Hectare.)
A farmer wishes to lease one Hectare of his land to an investor who wishes to use it to grow Tasmanian Blue Gum trees. The investor intends harvesting these fast growing trees and selling the wood to a paper making company as woodchip. The farmer, for his part, simply has to fence off suitable land for the investor to use. Rather than having to use new fencing around the whole area the farmer chooses a triangular site that allows existing fencing to be used on two sides (AB and AC in the diagram). The farmer measures the distance AB as 173 metres and measures ∠CAB as 40°. He wishes to locate point C so that ΔABC will have an area of one Hectare. He asks you to calculate the length AC for him. Calculate this length, rounding your answer **up** to the next whole metre.



The sine rule.

If we were given the triangle on the right and were asked to find a we could accurately draw the triangle and measure the required length. However, great accuracy is not easy to achieve and drawing can be time consuming. Once again we could proceed by drawing the perpendicular from B to AC so that our right triangle trigonometry work can be used, as shown below.





Draw the perpendicular from B to meet AC at D (see diag).

In ΔABD $\quad \sin 60° = \dfrac{h}{8}$

∴ $\quad h = 8 \sin 60°$

In ΔBCD $\quad \sin 50° = \dfrac{h}{a}$

∴ $\quad \sin 50° = \dfrac{8 \sin 60°}{a}$

Solving gives $\quad a \approx 9{\cdot}04$ cm

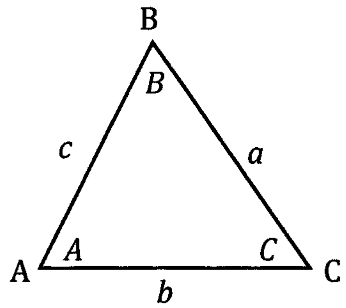
$\text{solve}\left(\sin(50) = \dfrac{8\times\sin(60)}{x}, x\right)$

$\{x = 9.044126999\}$

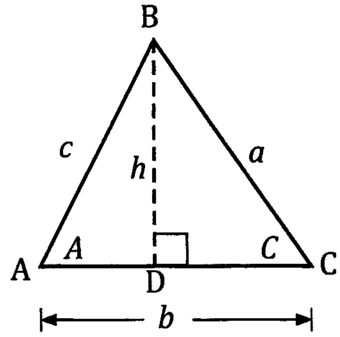
As we will see on the next page, if we apply this technique to a general triangle ABC we obtain the **sine rule**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Consider a triangle ABC as shown below left for an acute angled triangle and below right for an obtuse angled triangle.



Drawing the perpendicular from B to meet AC at D:

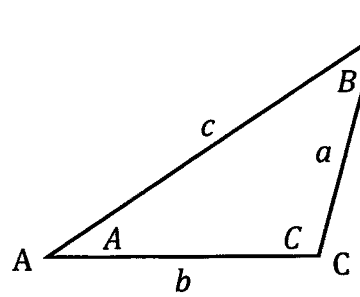


From ΔABD: $\sin A = \dfrac{h}{c}$

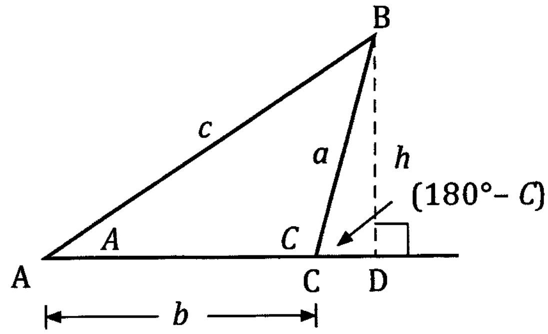
∴ $h = c \sin A$ ①

From ΔCBD: $\sin C = \dfrac{h}{a}$

∴ $h = a \sin C$ ②



Drawing the perpendicular from B to meet AC produced at D:



From ΔABD: $\sin A = \dfrac{h}{c}$

∴ $h = c \sin A$ ①

From ΔCBD: $\sin(180° - C) = \dfrac{h}{a}$

∴ $h = a \sin C$ ②

Thus for both the acute triangle and the obtuse triangle:

From ① and ② $c \sin A = a \sin C$

Thus $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ ③

If instead we draw the perpendicular from A to BC we obtain

$\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ ④

From ③ and ④ it follows that

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

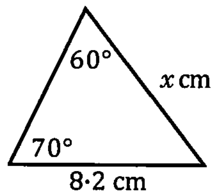
This is **the sine rule.**

→ Rather than learning this formula notice the pattern:
Any side on the sine of the opposite angle is equal to any other side on the sine of its opposite angle. ←

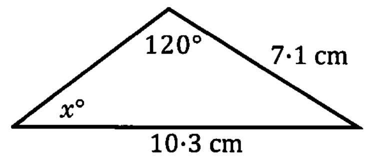
Example 4

Find the value of x in the following, giving answers correct to one decimal place.

(a)



(b)



(a) By the sine rule

$$\frac{x}{\sin 70°} = \frac{8{\cdot}2}{\sin 60°}$$

Multiply by sin 70° to isolate x

$$x = \frac{8{\cdot}2 \sin 70°}{\sin 60°}$$

$$= 8{\cdot}9 \text{ (to 1 dp)}$$

(b) By the sine rule

$$\frac{10{\cdot}3}{\sin 120°} = \frac{7{\cdot}1}{\sin x°}$$

Multiply by $(\sin x°)$ $(\sin 120°)$

$$10{\cdot}3 \sin x° = 7{\cdot}1 \sin 120°$$

$$\therefore \quad \sin x° \approx 0{\cdot}5970$$

$$x = 36{\cdot}7 \text{ (to 1 dp)}$$

Or, using the "solve" ability of some calculators:

$$\text{solve}\left(\frac{x}{\sin(70)} = \frac{8.2}{\sin(60)}, x\right)$$

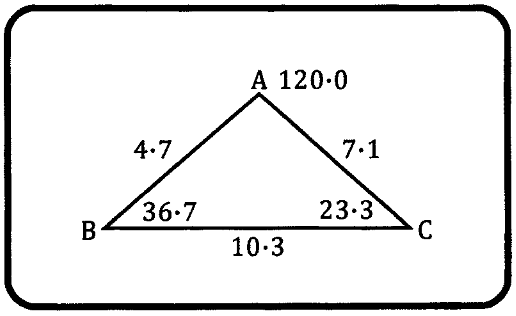
$\{x = 8.897521316\}$

$$\text{solve}\left(\frac{10.3}{\sin(120)} = \frac{7.1}{\sin(x)}, x\right) \Big| 0 \le x \le 180$$

$\{x = 143.346877, x = 36.65312298\}$

Note • In part (b) we went from $\sin x° \approx 0{\cdot}5970$ to $x = 36{\cdot}7$ (to 1 dp) despite there being another value of x between 0 and 180 for which $\sin x° = 0{\cdot}5970$, and that is $(180 - 36{\cdot}7)$, i.e. $143{\cdot}3$, as the calculator shows when asked for solutions in the interval $0 \le x \le 180$. However, in the given triangle x cannot be $143{\cdot}3$ because the triangle already has one obtuse angle and cannot have another. However, whilst this may not always be the case, and an *ambiguous* situation could occur when both answers are possible, this unit will not include such situations and sufficient information will be given to be able to dismiss one of the solutions.

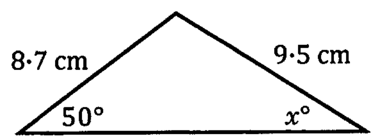
As was mentioned in the previous chapter on right triangles, some calculator programs allow the user to put in the known sides and angles of a triangle and, provided the information put in is sufficient, the program will determine the remaining sides and angles.



Also some calculators allow us to create a scale drawing of the triangle and find lengths and angles that way.
These programs can be useful but make sure that you understand the underlying idea of the sine rule (and the cosine rule which we will see later in this chapter) and can demonstrate the appropriate use of these rules when required to do so.

Example 5

Find the value of x in the triangle shown on the right.
(Give the answer correct to one decimal place.)



Note first that $x°$, being opposite a side of length 8·7 cm, must be less than the 50° which is opposite a side of length 9·5 cm. (For any two sides of a triangle, the larger of the two sides has the larger opposite angle.)

By the sine rule	$\frac{9 \cdot 5}{\sin 50°} = \frac{8 \cdot 7}{\sin x°}$
Multiply by (sin 50°) (sin $x°$)	$9 \cdot 5 \sin x° = 8 \cdot 7 \sin 50°$
∴	$\sin x° = \frac{8 \cdot 7 \sin 50°}{9 \cdot 5}$
Thus	$x \approx 44 \cdot 6$ (correct to 1 d.p.)

Or, using the "solve" ability of some calculators:

$$\text{solve}\left(\frac{9.5}{\sin(50)} = \frac{8.7}{\sin(x)}, x\right) \Big| 0 \le x \le 180$$

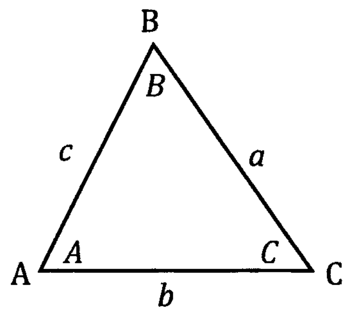
$$\{x = 135.4496775, x = 44.55032253\}$$

We then dismiss the obtuse angle because x had to be smaller than 50. (Or, had we not noticed this from the side lengths, we would reject the obtuse angle as the angle sum of the triangle would exceed 180°)

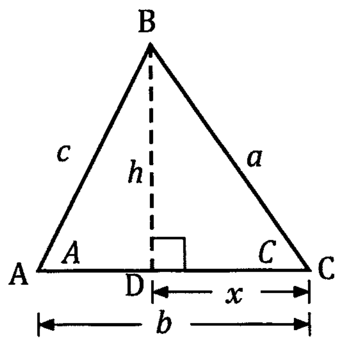
Thus, as before, $x = 44 \cdot 6$ (correct to 1 d.p.).

The cosine rule.

Again consider a triangle ABC as shown below left for an acute angled triangle and below right for an obtuse angled triangle. However, in this case, we use the fact that for cosines: $\cos(180° - C) = -\cos C$



Again we draw the perpendicular from B to meet AC at D:



From ΔCBD: $a^2 = h^2 + x^2$ ①

From ΔABD: $c^2 = h^2 + (b - x)^2$

$= h^2 + (b - x)(b - x)$

i.e. $c^2 = h^2 + b^2 + x^2 - 2bx$

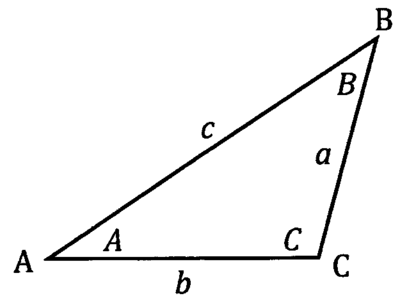
Using ①: $c^2 = a^2 + b^2 - 2bx$ ②

From ΔCBD: $\cos C = \dfrac{x}{a}$

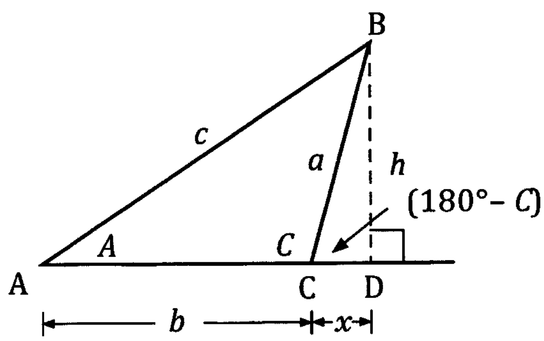
∴ $x = a\cos C$ ③

Using ② and ③ :

$c^2 = a^2 + b^2 - 2ab\cos C$



Again we draw the perpendicular from B to meet AC produced at D:



From ΔCBD: $a^2 = h^2 + x^2$ ①

From ΔABD: $c^2 = h^2 + (b + x)^2$

$= h^2 + (b + x)(b + x)$

i.e. $c^2 = h^2 + b^2 + x^2 + 2bx$

Using ①: $c^2 = a^2 + b^2 + 2bx$ ②

From ΔCBD: $\cos(180° - C) = \dfrac{x}{a}$

∴ $x = -a\cos C$ ③

Using ② and ③ :

$c^2 = a^2 + b^2 - 2ab\cos C$

Thus for both the acute triangle and the obtuse triangle $c^2 = a^2 + b^2 - 2ab\cos C$.

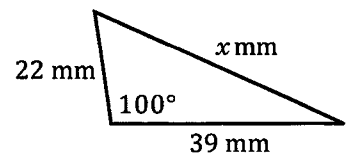
This is **the cosine rule:** $c^2 = a^2 + b^2 - 2ab\cos C$

Similarly $a^2 = b^2 + c^2 - 2bc\cos A$ and $b^2 = a^2 + c^2 - 2ac\cos B$

As was said with the sine rule, rather than learning the rule as a formula instead notice the pattern of what it is telling you:

→ The square of any side of a triangle is equal to the sum of the squares of the other two sides take away twice the product of the other two sides multiplied by the cosine of the angle between them. ←

Example 6
Find the value of x for the triangle shown sketched on the right.

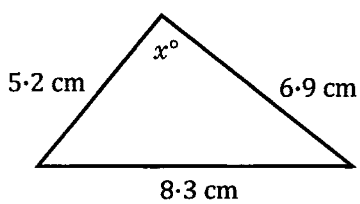


By the cosine rule:

$$x^2 = 22^2 + 39^2 - 2\,(22)\,(39)\cos 100°$$
$$\approx 2302{\cdot}98$$
$$x = 48 \quad \text{to the nearest integer.}$$

$22^2+39^2-2\times22\times39\times\cos(100)$
2302.980273
$\sqrt{}$ans
47.98937667

Example 7
Find the value of x for the triangle shown sketched on the right.



By the cosine rule:

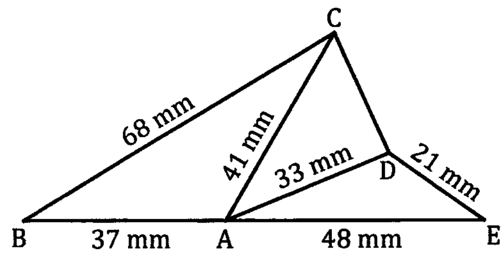
$$8{\cdot}3^2 = 5{\cdot}2^2 + 6{\cdot}9^2 - 2\,(5{\cdot}2)\,(6{\cdot}9)\cos x°$$
$$\cos x° = \frac{5{\cdot}2^2 + 6{\cdot}9^2 - 8{\cdot}3^2}{2\,(5{\cdot}2)\,(6{\cdot}9)}$$
$$\approx 0{\cdot}08027$$
$$x = 85 \quad \text{to the nearest integer.}$$

$\dfrac{5.2^2+6.9^2-8.3^2}{2\times5.2\times6.9}$
0.08026755853
$\cos^{-1}$(ans)
85.39605483

Note
- If you prefer to use the solve facility on your calculator make sure you can obtain the same answers as those shown.
- With the cosine rule, when solving equations of the form cos x = c we do not have to worry about there being two solutions in the range 0° to 180°. The cosine of an acute angle is positive whilst the cosine for an obtuse angle is negative. Thus an equation of the form cos x = c does <u>not</u> have two solutions for x in the range 0° to 180°. If c is positive the one solution will be an acute angle and if c is negative it will be an obtuse angle.

Example 8

The sketch on the right shows a system of three triangles with lengths and angles as indicated.
BAE is a straight line.
Find the length of CD.



Thoughts:
CD is one side of ΔACD. In this triangle we know the lengths of AC and AD so if we knew the size of ∠CAD we could apply the cosine rule to find the length of CD. We can find the size of ∠CAD if we first find the size of ∠CAB and the size of ∠DAE.

For ΔABC, applying the cosine rule:

$$68^2 = 41^2 + 37^2 - 2 \times 41 \times 37 \cos \angle BAC$$

$$\cos \angle BAC = \frac{41^2 + 37^2 - 68^2}{2 \times 41 \times 37}$$

$$\therefore \quad \angle BAC \approx 121{\cdot}3°$$

```
(41² + 37² - 68²) ÷ (2×41×37)
                 -0.5187870798
cos⁻¹ Ans
                   121.2509263
Ans → A
                   121.2509263
```

For ΔDAE, applying the cosine rule:

$$21^2 = 33^2 + 48^2 - 2 \times 33 \times 48 \cos \angle DAE$$

$$\cos \angle DAE = \frac{33^2 + 48^2 - 21^2}{2 \times 33 \times 48}$$

$$\therefore \quad \angle DAE \approx 21{\cdot}3°$$

```
(33² + 48² - 21²) ÷ (2×33×48)
                  0.9318181818
cos⁻¹ Ans
                   21.27996647
Ans → B
                   21.27996647
```

For ΔCAD, applying the cosine rule:

$$CD^2 = 33^2 + 41^2 - 2 \times 33 \times 41 \cos \angle CAD$$

$$\approx 622{\cdot}3$$

$$\therefore \quad CD \approx 24{\cdot}9$$

CD is of length 25 mm, to the nearest millimetre.

```
33²+41²-2×33×41cos(180-A-B)
                 622.297976
√Ans
                24.94590099
```

Notice from the calculator displays that the more accurate values for ∠BAC and ∠DAE were stored and later recalled for use, thus avoiding the risk of introducing unnecessary rounding errors.

Exercise 11C

The sine rule.

Given that each of the following equations are formed by applying the sine rule to an acute angled triangle solve for x, giving your answer correct to one decimal place in each case.

1. $\dfrac{x}{\sin 50°} = \dfrac{7{\cdot}3}{\sin 75°}$

2. $\dfrac{x}{\sin 32°} = \dfrac{12{\cdot}1}{\sin 78°}$

3. $\dfrac{12{\cdot}3}{\sin 60°} = \dfrac{x}{\sin 65°}$

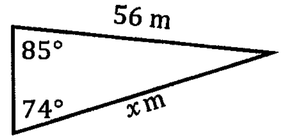
4. $\dfrac{8{\cdot}2}{\sin x°} = \dfrac{10}{\sin 85°}$

5. $\dfrac{7{\cdot}8}{\sin x°} = \dfrac{8{\cdot}3}{\sin 50°}$

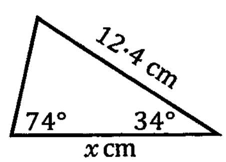
6. $\dfrac{6{\cdot}8}{\sin 50°} = \dfrac{7{\cdot}2}{\sin x°}$

Find the value of x in each of the following.

7.



8.



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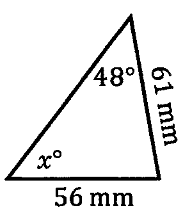
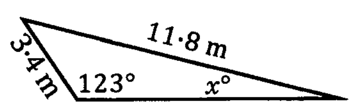


x is between 0 and 90.

10.



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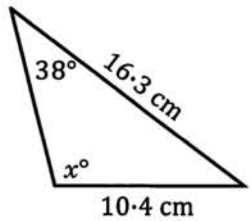
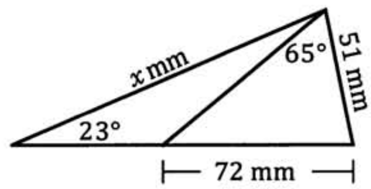


x is between 90 and 180.

12.



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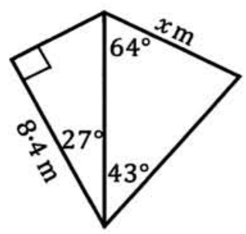

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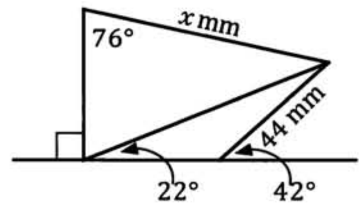
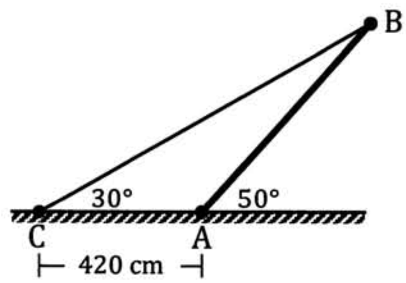
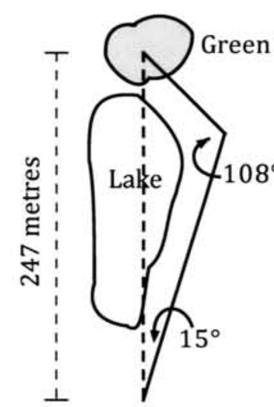


15. The diagram shows a pole AB with end A fixed on horizontal ground and the pole supported by a wire attached to end B and to a point C on the ground with AC = 420 centimetres.
The pole makes an angle of 50° with the ground and the wire makes an angle of 30° with the ground, as shown in the diagram.
Points A, B and C all lie in the same vertical plane.
Find the length of the pole giving your answer to the nearest centimetre.



16. Rather than risking the direct shot over a lake a golfer prefers to take two shots to get to the green as shown in the diagram on the right.
How much further is this two shot route than the direct route?



The cosine rule.

Given that each of the following equations are formed by applying the cosine rule to an acute angled triangle solve for x in each case, giving your answers correct to one decimal place.

17. $x^2 = 7^2 + 8^2 - 2\,(7)\,(8)\cos 56°$

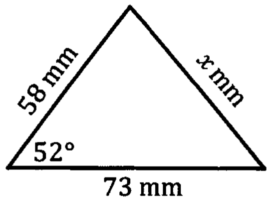
18. $x^2 = 3^2 + 2^2 - 2\,(3)\,(2)\cos 32°$

19. $3^2 = 5^2 + 7^2 - 2\,(5)\,(7)\cos x°$

20. $12^2 = 9^2 + 11^2 - 2\,(9)\,(11)\cos x°$

Find the value of x in each of the following.

21.



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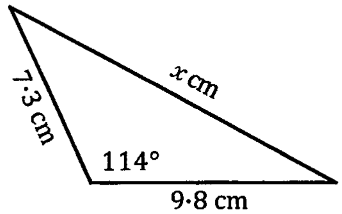

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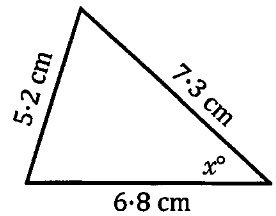

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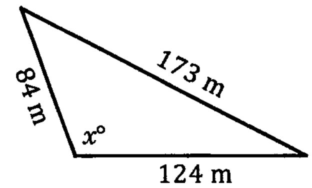

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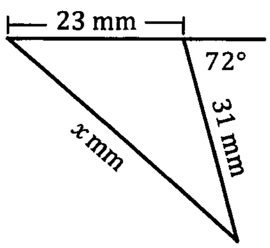

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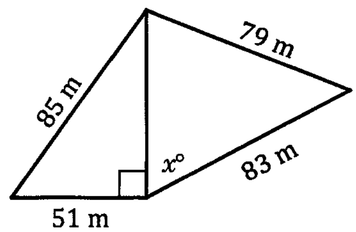

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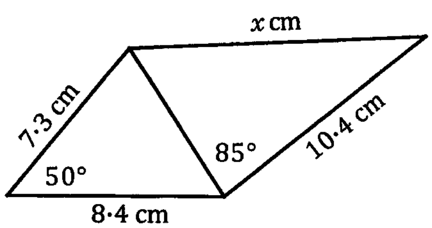

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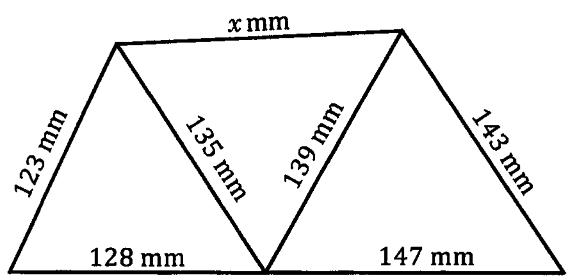
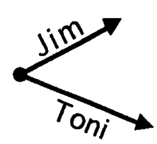


29. A boat travels 6·3 km due North and then turns 17° towards the West and travels a further 7·2 km. How far is it then from its initial position?

30. Jim and Toni leave the same point at the same time with Jim walking away at a speed of 1·4 m/s and Toni at a speed of 1·7 m/s, the two directions of travel making an angle of 50° with each other. If they both continue on these straight line paths how far are they apart after 8 seconds?

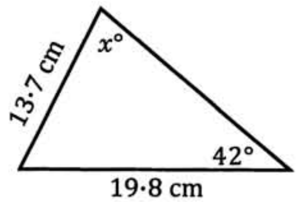


31. From location A, location B is 12·3 km away on a bearing of 070°.
From location A, location C is 7·2 km away on a bearing of 150°.
How far is B from C?

Miscellaneous.

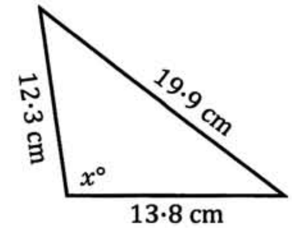
Find the value of x in each of the following.

32.



x is between 0 and 90.

33.



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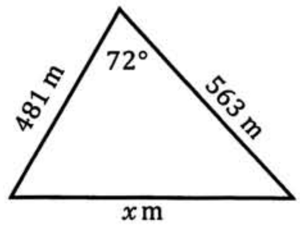

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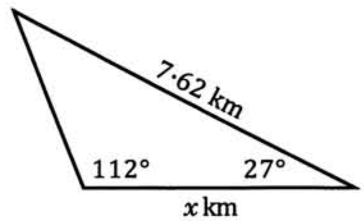

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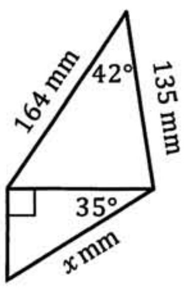

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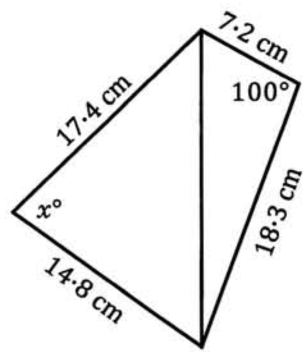

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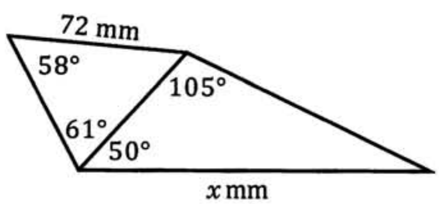

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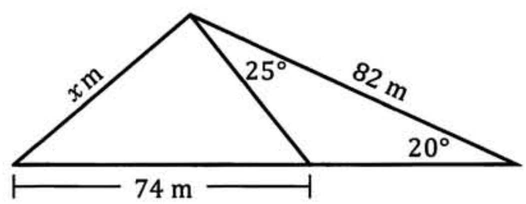
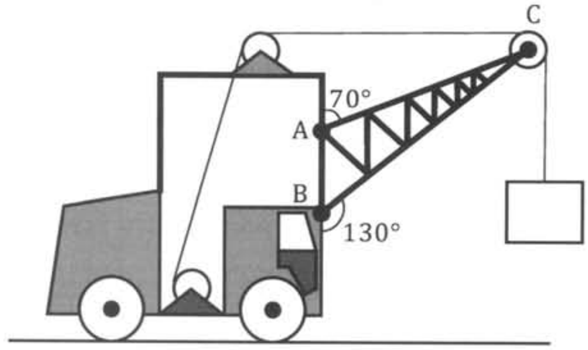


40. The diagram on the right shows a mobile crane used to lift containers from ships and transfer them to waiting container trucks. If AB is of length 300 centimetres find the lengths of AC and BC.

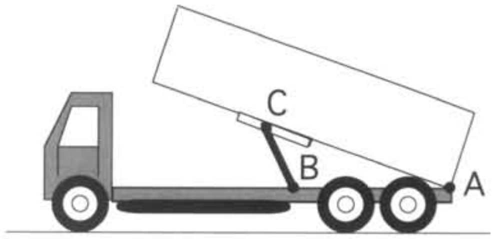


41. A triangle has sides of length 12·7 cm, 11·9 cm and 17·8 cm. Find the size of the smallest angle of the triangle, giving your answer to the nearest degree.

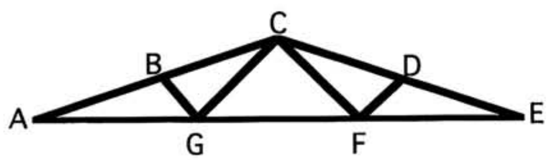
42. A parallelogram has sides of length 3·7 cm and 6·8 cm and the acute angle between the sides is 48°.
Find the lengths of the diagonals of the parallelogram.

43. The diagonals AC and BD, of parallelogram ABCD, intersect at E.
If ∠AED = 63° and the diagonals are of length 10·4 cm and 14·8 cm use the fact that the diagonals of a parallelogram bisect each other to determine the lengths of the sides of the parallelogram.

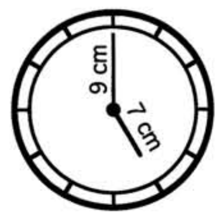
44. The tray of the tip truck shown on the right is tipped by the motor driving rod BC clockwise about B. As the tray tips, end C moves along the guide towards A.
If AB = 2 metres and BC = 1 metre find the size of ∠CAB when AC is
(a) 2·6 metres, (b) 2·1 metres.



45. The "W-type roof truss" shown on the right is to be constructed with
AE = 900 cm and AG = GF = FE.
∠DEF = 20°, ED = DC and the truss is symmetrical with the vertical line through C as the line of symmetry.
Calculate the following lengths, correct to the nearest cm
(a) CE, (b) ED, (c) DF, (d) CF.



46. Find, to the nearest millimetre, the distance between the tip of the 70 mm hour hand and the tip of the 90 mm minute hand of a clock at (a) 5 o'clock,
(b) 10 minutes past 5.

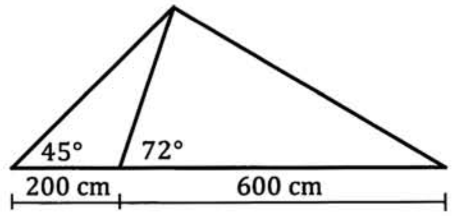


47. A coastal observation position is known to be 2·50 km from a lighthouse. The coastguard in the observation position is in radio and visual contact with a ship in distress at sea. If the coastguard looks towards the lighthouse and then towards the ship these two directions make an angle of 40° with each other. If the captain on the ship looks towards the observation position and then towards the lighthouse these two directions make an angle of 115° with each other. (The ship, the lighthouse and the observation position may all be assumed to be on the same horizontal level.)
How far is the ship from (a) the lighthouse,
(b) the coastal observation position?

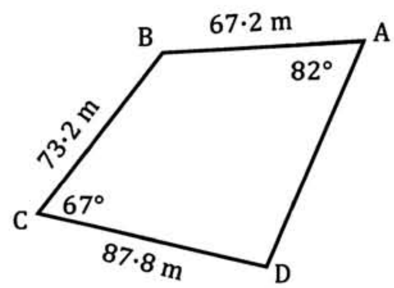
48. From a lighthouse, ship P is 7·3 km away on a bearing 070°.
    A second ship Q is on a bearing 150° from P and 130° from the lighthouse.
    (a) How far is Q from P?
    (b) How far is Q from the lighthouse?

49. From a lighthouse, ship A is 15·2 km away on a bearing 030° and ship B is 12·1 km away on a bearing 100°.
    How far, and on what bearing, is B from A?

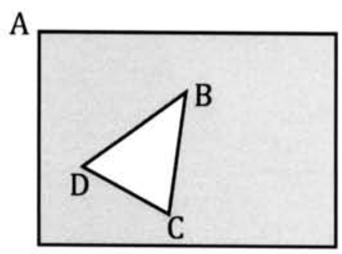
50. Ignoring any wastage needed for cutting, joining etc. what total length of steel would be needed to make twelve of the steel frameworks shown sketched on the right, rounding your answer up to the next ten metres.

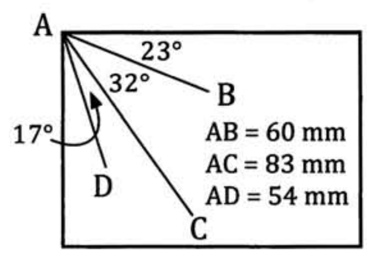


51. The diagram on the right shows the sketch made by a surveyor after taking measurements for a block of land ABCD.
    Find the area and the perimeter of the block.



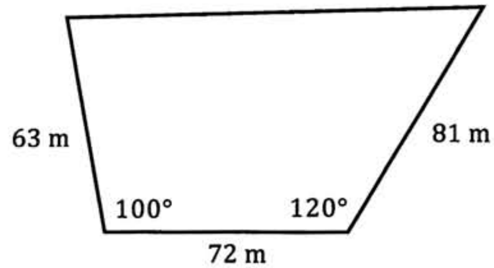
52. An engineering component consists of a rectangular metal plate with a triangular piece removed, as in the diagram below left.
    The removed piece is cut away by a computer controlled machine that is programmed to cut a triangle with vertices at the distances and angles shown on the diagram below right.





Find the area and the perimeter of the triangular piece that is removed.

53. The diagram on the right shows the sketch made by a surveyor after taking measurements for a block of land.
    Find the area of the block.

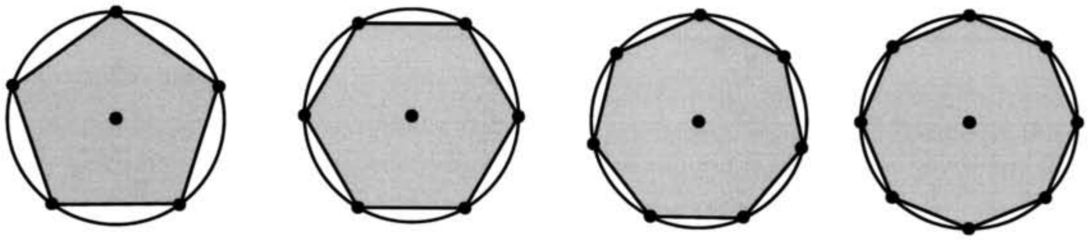


54. Make use of the cosine rule, and the rule for the area of a triangle given two sides and the included angle, to determine the area of a triangular block of land with sides of length 63 m, 22 m and 55 m and then check that your answer agrees with the following statement of the rule known as **Heron's rule**.
Area of a triangle with sides of length a, b and c is given by:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}.$$

Regular polygons.

Suppose that a regular n-sided polygon has all of its vertices touching the circumference of a circle of radius 1 unit.
For $n = 5, 6, 7$ and 8 this is shown below:



Find the area of each of the above polygons and investigate this situation for increasing integer values of n.

Miscellaneous Exercise Eleven.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Find the equation of the straight line passing through (1, 1) and (4, 7).
Determine which of the points listed below lie on this line.
A(7, 15), B(7, 13), C(2, 2), D(−1, 3), E(6, 11).

2. Solve the following equations.
(a) $3x = 51$ (b) $3x + 11 = 32$ (c) $2(3x + 2) - 5 = 11$
(d) $5 - 2(3x + 2) = 13$ (e) $\frac{x}{5} = 7$ (f) $\frac{x}{5} + 3 = 7$

3. Use the midpoint of each interval to estimate a mean for the following distribution of fifty scores.

Score	1 → 5	6 → 10	11 → 15	16 → 20	21 → 25	26 → 30	31 → 35
Frequency	8	14	9	7	6	4	2

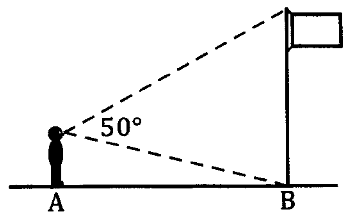
In what interval does the median score lie?

4. In a test the 12 boys in a class scored a mean of 23·4 and the 16 girls in the class scored a mean of 24·1. Find the mean of the whole class of 28 students.

5. I think of a number, double it, add three, multiply the answer by three and then add on twice the number I first thought of. If my final answer is one hundred and forty five what was the number I first thought of?

6. How long does it take an investment of \$2500 to grow to \$3220 in an account paying simple interest at the rate of 6·4% per annum?

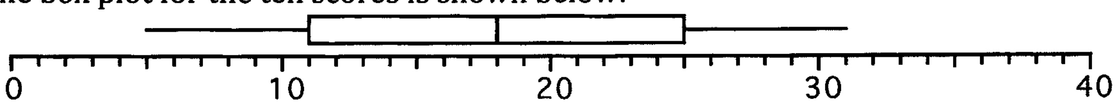
7. A and B are two points on level ground, 13 metres apart. A vertical flagpole at B subtends an angle of 50° at the eye of a person standing at A and whose "eye height" is 1·6 m (see diagram). Find the height of the flagpole.



8. Ten scores are shown below in ascending order, lowest score on the left:

   a − 1, d, 3d − 10, 2d − 1, a + 9, c + 2, c + 2, 6e + 7, c + 11, 7b − 4.

   The box plot for the ten scores is shown below:



   Find a, b, c, d and e and hence list the ten scores in ascending order.

9. In this chapter we have developed the sine and cosine rules and a formula for the area of a triangle so that we can determine the area and unknown side lengths and angle sizes of triangles that are not right angled. Could these rules be applied to right angled triangles? What happens when these formulae are applied to right angled triangles? Investigate.

10. Two boats leave a harbour at the same time.
    One travels due East at 7 km/hour and the other North-East at 5 km/hour.
    How far are the boats apart 90 minutes later, to the nearest 100 metres?

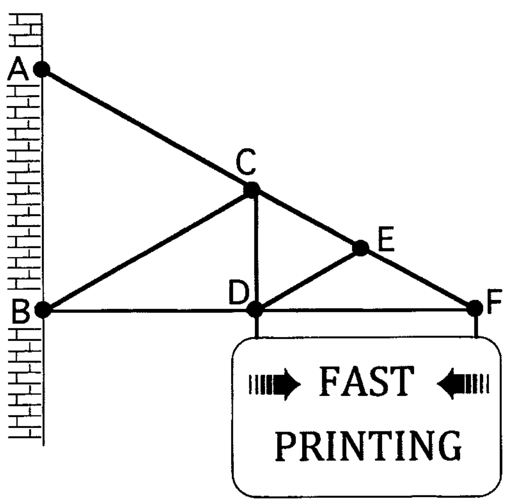
11. A print shop commissions a sign maker to make and install an advertising sign.

    The sign maker plans to suspend the sign from a framework as shown on the right.

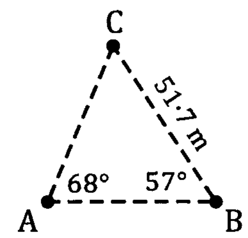
    The wall is vertical, DC is vertical and BF is horizontal. ΔBAC is equilateral, BA is of length 2 metres, ACF is a straight line and E is the midpoint of CF.

    Find the length of
    (a) AF (b) BF (c) CD (d) CE.

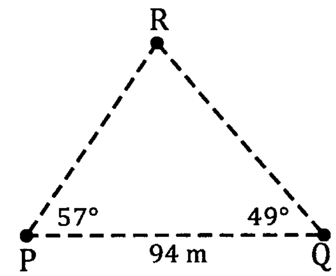
    (Give answers in metres, correct to two decimal places if rounding is necessary.)

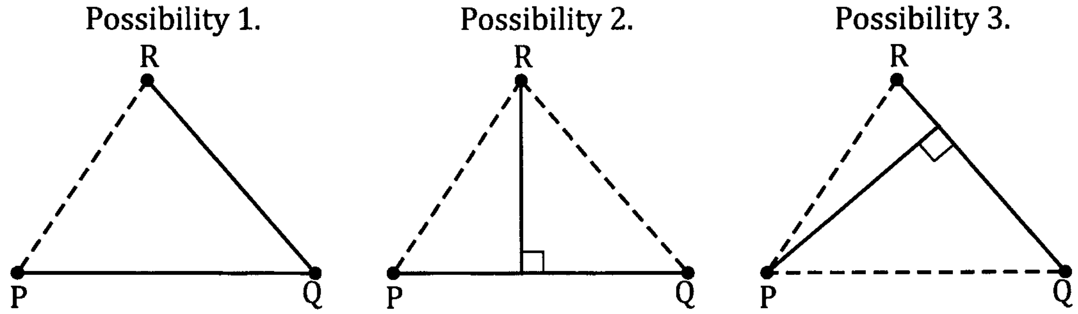


12. For the situation shown on the right how much shorter is the direct journey from A to C than the journey from A to C via B



13. Electrical cabling is to be installed to connect three locations, P, Q and R whose relative positions are as shown in the diagram on the right.
Direct connection from P to R is not feasible so three possibilities are considered as shown below:





Find the total length of each of these giving each answer to the nearest metre.

14. To provide cover for nurses absent due to sickness and other reasons, a health authority maintains a "pool" of nurses who are not attached to any particular ward or hospital but who can be directed to any particular area that is suffering a shortage. To assess how many nurses they need "in pool" they collect information regarding how many nurses are absent on each day for one year. The information is shown in the following table:

N°· of nurses absent	0 to 4	5 to 9	10 to 14	15 to 19	20 to 24	25 to 29	30 to 34
N°· of days	56	137	97	46	18	9	2

Use the interval midpoints to calculate the mean number of nurses absent per day over this 365 day period and determine the standard deviation of the distribution.
If the authority decides to have n nurses in pool where n is given by:

n = the next integer after (mean + 0·5 × standard deviation),

how many do they decide to have in pool?

15. The graphs below show the "population pyramids" of 2 countries, A and B.

	AGE DISTRIBUTION COUNTRY A			
Age	% of Popl	MALE	FEMALE	% of Popl
65+	4·1			5·0
55-64	4·3			4·6
45-54	5·2			5·2
35-44	7·1			7·0
25-34	8·3			8·3
15-24	8·2			8·3
5-14	8·7			8·6
0-4	3·6			3·5

AGE DISTRIBUTION COUNTRY B				
% of Popl	MALE	FEMALE	% of Popl	Age
2·5			1·8	65+
2·6			2·3	55-64
3·0			3·2	45-54
4·3			5·0	35-44
6·2			6·5	25-34
8·2			8·6	15-24
15·0			14·3	5-14
8·2			8·3	0-4

(a) In each pyramid the shaded bar at the bottom is smaller than the one above it? Suggest a reason why this might be.

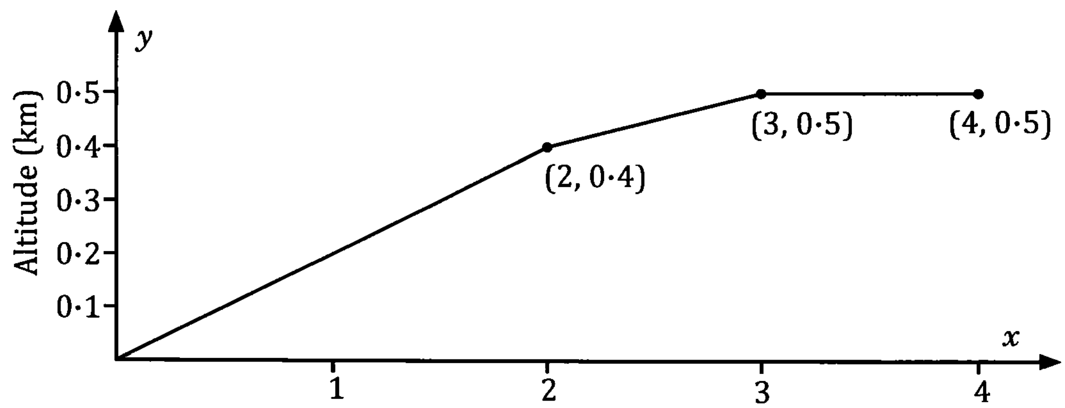
(b) For which of the age ranges does A have more males than females?

(c) For which of the age ranges does B have more females than males?

(d) Country B has a population of 52 000 000. How many people aged 55 and over does the country have?

(e) Which of the two countries do you think is the more highly developed? Give reasons for your choice.

16. The diagram below shows the approximate path of an aircraft from take off to a point 4 km horizontally from take off drawn as three straight line sections.



Horizontal distance from point of take off (km).

Find the equation of each of the three lines giving your answer in the form:

For x from 0 to 2 : $y = \ldots$

For x from 2 to ... : $y = \ldots$

For x ..... : $y = \ldots$